

ON THE D-AFFINITY OF FLAG VARIETIES IN POSITIVE CHARACTERISTIC

ALEXANDER SAMOKHIN

ABSTRACT. Let \mathbf{G} be a simple simply connected algebraic group of type \mathbf{B}_2 over an algebraically closed field k of odd characteristic. We prove that the flag variety \mathbf{G}/\mathbf{B} is D-affine. This extends an earlier result of Andersen and Kaneda [2].

1. Introduction

Let X be a smooth algebraic variety over an algebraically closed field k , and let \mathcal{D}_X be the sheaf of differential operators on X . Then X is said to be D-affine if the following two conditions hold: (i) for any \mathcal{D}_X -module M that is quasi-coherent over \mathcal{O}_X the natural morphism $\mathcal{D}_X \otimes_{\Gamma(\mathcal{D}_X)} \Gamma(M) \rightarrow M$ is onto, and (ii) $H^i(X, \mathcal{D}_X) = 0$ for $i > 0$. Let \mathbf{G} be a semisimple algebraic group over k and \mathbf{P} a parabolic subgroup of \mathbf{G} . If k is of characteristic zero then the well-known Beilinson–Bernstein localization theorem [3] states that \mathbf{G}/\mathbf{P} is D-affine. Much less is known when k is of characteristic $p > 0$. Haastert [4] showed that projective spaces and the flag variety of the group \mathbf{SL}_3 are D-affine, and Langer [7] proved the D-affinity for odd-dimensional quadrics if the characteristic of k is greater than the dimension of variety (while even-dimensional quadrics turn out to be not D-affine). Even earlier, Kashiwara and Lauritzen [6] produced a counterexample to the D-affinity: their result implies that the flag variety of the group \mathbf{SL}_5 is not D-affine in any characteristic. Nevertheless, the question about which flag varieties are D-affine in positive characteristic remains open; nothing was known except the above cases. In the present paper we show that the flag variety of the group \mathbf{Sp}_4 is D-affine in odd characteristic. By [4], it is sufficient to prove that $H^i(\mathbf{Sp}_4/\mathbf{B}, \mathcal{D}_{\mathbf{Sp}_4/\mathbf{B}}) = 0$ for $i > 0$. This is achieved by showing that all the terms of the p -filtration on the sheaf $\mathcal{D}_{\mathbf{Sp}_4/\mathbf{B}}$ have vanishing higher cohomology groups, thus extending an earlier result of Andersen and Kaneda [2], where they showed the cohomology vanishing of the first term of the p -filtration (in any characteristic). Contrary to their representation theoretic approach, we use simple geometric arguments to reduce the problem to computing cohomology groups of line bundles on the flag variety \mathbf{Sp}_4/\mathbf{B} . Cohomology of line bundles on flag varieties in the rank two case are well understood thanks to Andersen’s et al. work (see [1] for a recent survey and [5] for a comprehensive treatment); working out the cohomology groups in question completes the proof. However, for the sake of consistency with our approach and convenience of the reader, we explicitly show all the necessary vanishings without the use of general theory.

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2. Preliminaries

Let k be an algebraically closed field of odd characteristic $p > 0$, and V be a symplectic vector space of dimension 4 over k . Let G be the symplectic group Sp_4 over k ; the root system of G is of type B_2 . Let B be a Borel subgroup of G . Consider the flag variety G/B . The group G has two parabolic subgroups P_α and P_β that correspond to the simple roots α and β , the root β being the long root. The homogeneous spaces G/P_α and G/P_β are isomorphic to the 3-dimensional quadric Q_3 and \mathbb{P}^3 , respectively. Denote q and π the two projections of G/B onto Q_3 and \mathbb{P}^3 . The line bundles on G/B that correspond to the fundamental weights ω_α and ω_β are isomorphic to $\pi^*\mathcal{O}_{\mathbb{P}^3}(1)$ and $q^*\mathcal{O}_{Q_3}(1)$, respectively. The canonical line bundle $\omega_{G/B}$ corresponds to the weight $-2\rho = -2(\omega_\alpha + \omega_\beta)$ and is isomorphic to $\pi^*\mathcal{O}_{\mathbb{P}^3}(-2) \otimes q^*\mathcal{O}_{Q_3}(-2)$. The projection π is the projective bundle over \mathbb{P}^3 associated to a rank two vector bundle N over $\mathbb{P}^3 = \mathbb{P}(V)$, and the projection q is the projective bundle associated to the spinor bundle \mathcal{U}_2 on Q_3 . The bundle N is symplectic, that is there is a non-degenerate skew-symmetric pairing $\wedge^2 N \rightarrow \mathcal{O}_{\mathbb{P}^3}$ that is induced by the given symplectic structure on V . There is a short exact sequence on \mathbb{P}^3 :

$$(1) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow \Omega_{\mathbb{P}^3}^1(1) \rightarrow N \rightarrow 0,$$

while the spinor bundle \mathcal{U}_2 , which is also isomorphic to the restriction of the rank two universal bundle on $Gr_{2,4} = Q_4$ to Q_3 , fits into a short exact sequence on Q_3 :

$$(2) \quad 0 \rightarrow \mathcal{U}_2 \rightarrow V \otimes \mathcal{O}_{Q_3} \rightarrow \mathcal{U}_2^* \rightarrow 0.$$

Let $\mathcal{D}_{G/B}$ be the sheaf of differential operators on G/B . By Theorem 4.4.1 of [4] flag varieties are quasi D-affine, that is every \mathcal{D} -module on a flag variety is \mathcal{D} -generated by its global sections. This implies that the D-affinity of G/B will follow if the sheaf $\mathcal{D}_{G/B}$ has vanishing higher cohomology groups. The main result of the paper is the following theorem:

Theorem 2.1.

$$H^i(G/B, \mathcal{D}_{G/B}) = 0$$

for $i > 0$.

Proof. Let $F^n : G/B \rightarrow G/B$ be the n -th absolute Frobenius morphism. By Theorem 1.2.4 of [4] there is an isomorphism of sheaves $\mathcal{D}_{G/B} = \bigcup_{n \geq 1} \mathcal{E}nd(F_*^n \mathcal{O}_{G/B})$. Fix $i \geq 0$. Clearly, $H^i(G/B, \mathcal{E}nd(F_*^n \mathcal{O}_{G/B})) = 0$ for all $n \geq 1$ implies $H^i(G/B, \mathcal{D}_{G/B}) = 0$. The statement will follow from Theorem 2.2 below, whose proof occupies the next two sections. \square

Theorem 2.2.

$$H^i(G/B, \mathcal{E}nd(F_*^n \mathcal{O}_{G/B})) = 0$$

for $i > 0$ and $n \geq 1$.

The method used in [4] and [2] was to identify the sheaf $\mathcal{E}nd(F_*^n \mathcal{O}_{G/B})$ with an equivariant vector bundle on G/B associated to the induced module $\text{Ind}_{B^n}^{G_n B}(2(p^n - 1)\rho)$, where G_n is the n -th Frobenius kernel, and to study an appropriate filtration on such a module. Our main tool is a short exact sequence from [11] that relates the Frobenius pushforwards of the structure sheaves on the total space of a \mathbb{P}^1 -bundle and on the base variety.

3. Proof of Theorem 2.2

Recall the short exact sequence from [11] mentioned above. Assume given a smooth variety S and a locally free sheaf \mathcal{E} of rank 2 on S . Let $X = \mathbb{P}_S(\mathcal{E})$ be the projective bundle over S and $\pi : X \rightarrow S$ the projection. Denote $\mathcal{O}_\pi(-1)$ the relative invertible sheaf. One has $\pi_*\mathcal{O}_\pi(1) = \mathcal{E}^*$.

Lemma 3.1. *For any $n \geq 1$ there is a short exact sequence of vector bundles on X :*

$$(3) \quad 0 \rightarrow \pi^*F_*^n\mathcal{O}_S \rightarrow F_*^n\mathcal{O}_X \rightarrow \pi^*(F_*^n(D^{p^n-2}\mathcal{E} \otimes \det \mathcal{E}) \otimes \det \mathcal{E}^*) \otimes \mathcal{O}_\pi(-1) \rightarrow 0.$$

Here $D^k\mathcal{E} = (S^k\mathcal{E}^*)^*$ is the k -th divided power of \mathcal{E} .

For convenience of the reader we recall the proof.

Proof. Let $D^b(X)$ be the bounded derived category of coherent sheaves on X , and denote $[1]$ the shift functor. By [8], for any object $A \in D^b(X)$ there is a distinguished triangle:

$$(4) \quad \cdots \rightarrow \pi^*R^\bullet\pi_*A \rightarrow A \rightarrow \pi^*(\tilde{A}) \otimes \mathcal{O}_\pi(-1) \rightarrow \pi^*R^\bullet\pi_*A[1] \rightarrow \cdots$$

The object \tilde{A} can be found by tensoring the triangle (4) with $\mathcal{O}_\pi(-1)$ and applying the functor $R^\bullet\pi_*$ to the obtained triangle. Given that $R^\bullet\pi_*\mathcal{O}_\pi(-1) = 0$, we get an isomorphism:

$$(5) \quad R^\bullet\pi_*(A \otimes \mathcal{O}_\pi(-1)) \simeq \tilde{A} \otimes R^\bullet\pi_*\mathcal{O}_\pi(-2).$$

One has $R^\bullet\pi_*\mathcal{O}_\pi(-2) = \det \mathcal{E}[-1]$. Tensoring both sides of the isomorphism (5) with $\det \mathcal{E}^*$, we get:

$$(6) \quad \tilde{A} = R^\bullet\pi_*(A \otimes \mathcal{O}_\pi(-1)) \otimes \det \mathcal{E}^*[1].$$

Let now A be the vector bundle $F_*^n\mathcal{O}_X$. The triangle (4) becomes in this case:

$$(7) \quad \cdots \rightarrow \pi^*R^\bullet\pi_*F_*^n\mathcal{O}_X \rightarrow F_*^n\mathcal{O}_X \rightarrow \pi^*(\tilde{A}) \otimes \mathcal{O}_\pi(-1) \rightarrow \pi^*R^\bullet\pi_*F_*^n\mathcal{O}_X[1] \rightarrow \cdots$$

where $\tilde{A} = R^\bullet\pi_*(F_*^n\mathcal{O}_X \otimes \mathcal{O}_\pi(-1)) \otimes \det \mathcal{E}^*[1]$. Recall that for a coherent sheaf \mathcal{F} on X one has an isomorphism $R^i\pi_*F_*^n\mathcal{F} = F_*^nR^i\pi_*\mathcal{F}$, the Frobenius morphism being finite and commuting with arbitrary morphisms. Therefore,

$$(8) \quad R^\bullet\pi_*F_*^n\mathcal{O}_X = F_*^nR^\bullet\pi_*\mathcal{O}_X = F_*^n\mathcal{O}_S.$$

On the other hand, by the projection formula one has $R^\bullet\pi_*(F_*^n\mathcal{O}_X \otimes \mathcal{O}_\pi(-1)) = R^\bullet\pi_*(F_*^n\mathcal{O}_\pi(-p^n)) = F_*^nR^\bullet\pi_*\mathcal{O}_\pi(-p^n)$. The relative Serre duality for π gives:

$$(9) \quad R^\bullet\pi_*\mathcal{O}_\pi(-p^n) = D^{p^n-2}\mathcal{E} \otimes \det \mathcal{E}[-1].$$

Let $\tilde{\mathcal{E}}$ be the vector bundle $D^{p^n-2}\mathcal{E} \otimes \det \mathcal{E}$. Putting these isomorphisms together we see that the triangle (7) can be rewritten as follows:

$$(10) \quad \cdots \rightarrow \pi^*F_*^n\mathcal{O}_S \rightarrow F_*^n\mathcal{O}_X \rightarrow \pi^*(F_*^n\tilde{\mathcal{E}} \otimes \det \mathcal{E}^*) \otimes \mathcal{O}_\pi(-1) \xrightarrow{[1]} \cdots$$

Therefore, the above distinguished triangle is in fact a short exact sequence of vector bundles on X :

$$(11) \quad 0 \rightarrow \pi^*F_*^n\mathcal{O}_S \rightarrow F_*^n\mathcal{O}_X \rightarrow \pi^*(F_*^n\tilde{\mathcal{E}} \otimes \det \mathcal{E}^*) \otimes \mathcal{O}_\pi(-1) \rightarrow 0.$$

□

We will use the projection $q : \mathbf{G}/\mathbf{B} \rightarrow \mathbf{Q}_3$ to compute the bundle $F_*^n \mathcal{O}_{\mathbf{G}/\mathbf{B}}$. Applying Lemma 3.1 in this case, we get a short exact sequence:

$$(12) \quad 0 \rightarrow q^* F_*^n \mathcal{O}_{\mathbf{Q}_3} \rightarrow F_*^n \mathcal{O}_{\mathbf{G}/\mathbf{B}} \rightarrow q^* (F_*^n (\mathcal{D}^{p^n-2} \mathcal{U}_2(-1)) \otimes \mathcal{O}_{\mathbf{Q}_3}(1)) \otimes \mathcal{O}_q(-1) \rightarrow 0.$$

Here $\mathcal{O}_q(-1) = \pi^* \mathcal{O}_{\mathbb{P}^3}(-1)$ is the relative line bundle with respect to the projection q . Apply the functor $\text{Hom}(-, F_*^n \mathcal{O}_{\mathbf{G}/\mathbf{B}})$ to this sequence. Consider first the groups $\text{Ext}^i(q^* F_*^n \mathcal{O}_{\mathbf{Q}_3}, F_*^n \mathcal{O}_{\mathbf{G}/\mathbf{B}})$. By adjunction we get an isomorphism:

$$(13) \quad \text{Ext}^i(q^* F_*^n \mathcal{O}_{\mathbf{Q}_3}, F_*^n \mathcal{O}_{\mathbf{G}/\mathbf{B}}) = \text{Ext}^i(F_*^n \mathcal{O}_{\mathbf{Q}_3}, F_*^n \mathcal{O}_{\mathbf{Q}_3}).$$

Indeed, $R^\bullet q_* F_*^n \mathcal{O}_{\mathbf{G}/\mathbf{B}} = F_*^n R^\bullet q_* \mathcal{O}_{\mathbf{G}/\mathbf{B}} = F_*^n \mathcal{O}_{\mathbf{Q}_3}$.

Lemma 3.2. $\text{Ext}^i(F_*^n \mathcal{O}_{\mathbf{Q}_3}, F_*^n \mathcal{O}_{\mathbf{Q}_3}) = 0$ for $i > 0$ and $n \geq 1$.

Proof. For $n = 1$ this follows from [9]. For quadrics of arbitrary dimension an explicit decomposition of the Frobenius pushforward of a line bundle was found in [7]; in particular, this implies Lemma 3.2. However, it is worth giving an independent proof that is based on the argument from [9]; the proof of Theorem 2.2 is just an extension of it. Recall (Lemma 2.3, *loc.cit.*) that there is an isomorphism of cohomology groups:

$$(14) \quad \text{Ext}^i(F_*^n \mathcal{O}_{\mathbf{Q}_3}, F_*^n \mathcal{O}_{\mathbf{Q}_3}) = H^i(\mathbf{Q}_3 \times \mathbf{Q}_3, (F^n \times F^n)^*(i_* \mathcal{O}_\Delta) \otimes (\mathcal{O}_{\mathbf{Q}_3} \boxtimes \omega_{\mathbf{Q}_3}^{1-p^n}))$$

There is a resolution of the sheaf $i_* \mathcal{O}_\Delta$ (Lemma 3.1, [9]):

$$(15) \quad 0 \rightarrow \mathcal{U}_2 \boxtimes \mathcal{U}_2(-2) \rightarrow \Psi_2 \boxtimes \mathcal{O}_{\mathbf{Q}_3}(-2) \rightarrow \Psi_1 \boxtimes \mathcal{O}_{\mathbf{Q}_3}(-1) \rightarrow \mathcal{O}_{\mathbf{Q}_3} \boxtimes \mathcal{O}_{\mathbf{Q}_3} \rightarrow i_* \mathcal{O}_\Delta \rightarrow 0,$$

This is a particular case of Kapranov's resolution of the diagonal for quadrics. Put $\Psi_0 = \mathcal{O}_{\mathbf{Q}_3}$ and $\Psi_3 = \mathcal{U}_2$. When k is of characteristic zero, the bundles Ψ_i for $i = 1, 2$ can explicitly be described as follows: the bundle Ψ_1 is isomorphic to the restriction of $\Omega^1(1)$ on \mathbb{P}^4 to \mathbf{Q}_3 , the quadric \mathbf{Q}_3 being naturally embedded into $\mathbb{P}^4 = \mathbb{P}(\mathbf{W})$, and the bundle Ψ_2 fits into the short exact sequence

$$(16) \quad 0 \rightarrow \Omega_{\mathbb{P}^4}^2(2) \otimes \mathcal{O}_{\mathbf{Q}_3} \rightarrow \Psi_2 \rightarrow \mathcal{O}_{\mathbf{Q}_3} \rightarrow 0.$$

Let us check that the same descriptions of Ψ_1 and Ψ_2 are valid when the characteristic of k is an odd prime. This amounts to computation of cohomology groups. Indeed, for any coherent sheaf \mathcal{E} on \mathbf{Q}_3 there is a standard spectral sequence converging to \mathcal{E} , and whose E_1 -term is equal to $H^i(\mathbf{Q}_3, \mathcal{E} \otimes \mathcal{O}_{\mathbf{Q}_3}(j)) \otimes \Psi_{-j}$ for $j = -2, \dots, 0$, and $H^i(\mathbf{Q}_3, \mathcal{E} \otimes \mathcal{U}_2(-2)) \otimes \Psi_{-3}$ for $j = -3$. Taking \mathcal{E} to be $\Omega_{\mathbb{P}^4}^1(1) \otimes \mathcal{O}_{\mathbf{Q}_3}$ or $\Omega_{\mathbb{P}^4}^2(2) \otimes \mathcal{O}_{\mathbf{Q}_3}$ and computing the terms of spectral sequence we see that the cohomology groups in question are the same as in characteristic zero, thus arriving at the above resolutions for Ψ_1 and Ψ_2 .

Arguing as in the proof of Theorem 3.2 of [9], we conclude that Lemma 3.2 follows from the following statement:

Proposition 3.1. $H^i(\mathbf{Q}_3, F^n^* \mathcal{U}_2) = 0$ for $i \neq 2$ and $n \geq 1$.

□

Proof. Denote $\mathcal{O}_\pi(-1)$ the relative line bundle with respect to the projection $\pi : \mathbf{G}/\mathbf{B} = \mathbb{P}(\mathbf{N}) \rightarrow \mathbf{G}/\mathbf{P}_\alpha = \mathbb{P}^3$. Consider the short exact sequence

$$(17) \quad 0 \rightarrow \pi^* \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow q^* \mathcal{U}_2 \rightarrow \mathcal{O}_\pi(-1) \rightarrow 0.$$

Applying the functor F^{n*} to it we get:

$$(18) \quad 0 \rightarrow \pi^* \mathcal{O}_{\mathbb{P}^3}(-p^n) \rightarrow q^* F^{n*} \mathcal{U}_2 \rightarrow \mathcal{O}_{\pi}(-p^n) \rightarrow 0.$$

First, one has $H^i(\mathbf{G}/\mathbf{B}, \pi^* \mathcal{O}_{\mathbb{P}^3}(-p^n)) = H^i(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-p^n)) = 0$ for $i \neq 3$. Let us show that $H^i(\mathbf{G}/\mathbf{B}, \mathcal{O}_{\pi}(-p^n)) = 0$ for $i < 2$. Indeed, $H^0(\mathbf{G}/\mathbf{B}, \mathcal{O}_{\pi}(-p^n)) = 0$, the line bundle $\mathcal{O}_{\pi}(-k)$ being not effective for any k . Let us consider H^1 . One has $R^\bullet \pi_* \mathcal{O}_{\pi}(-p^n) = D^{p^n-2} \mathbf{N}[-1]$. Thus, $H^1(\mathbf{G}/\mathbf{B}, \mathcal{O}_{\pi}(-p^n)) = H^0(\mathbb{P}^3, D^{p^n-2} \mathbf{N})$. For any $k > 0$ there is a short exact sequence on \mathbf{G}/\mathbf{B} :

$$(19) \quad 0 \rightarrow \mathcal{O}_{\pi}(-k) \rightarrow \pi^* D^k \mathbf{N} \rightarrow \pi^* D^{k-1} \mathbf{N} \otimes \mathcal{O}_{\pi}(1) \rightarrow 0.$$

It is obtained from the relative Euler sequence

$$(20) \quad 0 \rightarrow \mathcal{O}_{\pi}(-1) \rightarrow \pi^* \mathbf{N} \rightarrow \mathcal{O}_{\pi}(1) \rightarrow 0$$

by taking first its k -th symmetric power and then passing to the dual (since the bundle \mathbf{N} is symplectic, it is self-dual, that is $\mathbf{N} = \mathbf{N}^*$). We saw above that the line bundle $\mathcal{O}_{\pi}(-k)$ did not have global sections. Using the sequence (19) for $k = p^n - 2, p^n - 3, \dots, 1$ and descending induction, we see that $H^0(\mathbf{G}/\mathbf{B}, \pi^* D^k \mathbf{N}) = H^0(\mathbb{P}^3, D^k \mathbf{N}) = 0$. This implies $H^i(\mathbf{Q}_3, F^{n*} \mathcal{U}_2) = 0$ for $i < 2$. By Serre duality $H^3(\mathbf{Q}_3, F^{n*} \mathcal{U}_2) = H^0(\mathbf{Q}_3, F^{n*} \mathcal{U}_2^* \otimes \omega_{\mathbf{Q}_3})^*$. Recall that $\omega_{\mathbf{Q}_3} = \mathcal{O}_{\mathbf{Q}_3}(-3)$. Dualizing the sequence (18) and tensoring it with $\omega_{\mathbf{Q}_3}$, we see that the bundle $F^{n*} \mathcal{U}_2^* \otimes \omega_{\mathbf{Q}_3}$ is an extension of two line bundles, both of which are non-effective. Thus, $H^3(\mathbf{Q}_3, F^{n*} \mathcal{U}_2) = 0$. Finally, one gets a short exact sequence:

$$(21) \quad 0 \rightarrow H^2(\mathbf{Q}_3, F^{n*} \mathcal{U}_2) \rightarrow H^2(\mathbf{G}/\mathbf{B}, \mathcal{O}_{\pi}(-p^n)) \rightarrow H^3(\mathbf{G}/\mathbf{B}, \pi^* \mathcal{O}(-p^n)) \rightarrow 0,$$

and the statement follows.

Remark 3.1. *The (non)-vanishing of the first cohomology group of a line bundle on arbitrary flag variety was completely determined by H.H.Andersen (cf. [1], 2.3). The line bundle $\mathcal{L}_{\chi} = \mathcal{O}_{\pi}(-p^n)$ corresponds to the weight $\chi = p^n \omega_{\alpha} - p^n \omega_{\beta}$. Using Andersen's criterion one immediately checks the vanishing of $H^1(\mathbf{G}/\mathbf{B}, \mathcal{L}_{\chi})$.*

□

Next step is the following vanishing:

Lemma 3.3.

$$(22) \quad \text{Ext}^i(q^*(F_*^n(D^{p^n-2} \mathcal{U}_2(-1)) \otimes \mathcal{O}_{\mathbf{Q}_3}(1)) \otimes \mathcal{O}_q(-1), F_*^n \mathcal{O}_{\mathbf{G}/\mathbf{B}}) = 0$$

for $i > 0$ and $n \geq 1$.

Clearly, Lemma 3.2 and Lemma 3.3 will imply Theorem 2.2.

Proof. By the projection formula, one has:

$$(23) \quad \begin{aligned} \text{Ext}^i(q^*(F_*^n(D^{p^n-2} \mathcal{U}_2(-1)) \otimes \mathcal{O}_{\mathbf{Q}_3}(1)) \otimes \mathcal{O}_q(-1), F_*^n \mathcal{O}_{\mathbf{G}/\mathbf{B}}) = \\ = \text{Ext}^i(F_*^n(D^{p^n-2} \mathcal{U}_2(-1)) \otimes \mathcal{O}_{\mathbf{Q}_3}(1), F_*^n S^{p^n} \mathcal{U}_2^*). \end{aligned}$$

Indeed,

$$(24) \quad \begin{aligned} R^\bullet q_*(F_*^n \mathcal{O}_{\mathbf{G}/\mathbf{B}} \otimes \mathcal{O}_q(1)) &= R^\bullet q_*(F_*^n \mathcal{O}_{\mathbf{G}/\mathbf{B}} \otimes \pi^* \mathcal{O}_{\mathbb{P}^3}(1)) = R^\bullet q_* F_*^n \pi^* \mathcal{O}_{\mathbb{P}^3}(p^n) = \\ &= F_*^n R^\bullet q_* \pi^* \mathcal{O}_{\mathbb{P}^3}(p^n) = F_*^n S^{p^n} \mathcal{U}_2^*. \end{aligned}$$

Recall that a right adjoint functor to F_*^n on a smooth variety X over k is given by the formula:

$$(25) \quad F^{n!}(\cdot) = F^{n*}(\cdot) \otimes \omega_X^{1-p^n}.$$

Therefore,

$$(26) \quad \begin{aligned} \text{Ext}^i(F_*^n(D^{p^n-2}\mathcal{U}_2(-1)) \otimes \mathcal{O}_{Q_3}(1), F_* S^{p^n}\mathcal{U}_2^*) &= \\ = \text{Ext}^i(D^{p^n-2}\mathcal{U}_2(-1), F^{n*}F_*^n S^{p^n}\mathcal{U}_2^* \otimes \mathcal{O}_{Q_3}(-p^n) \otimes \omega_{Q_3}^{1-p^n}). \end{aligned}$$

We have $\mathcal{O}_{Q_3}(-p^n) \otimes \omega_{Q_3}^{1-p^n} = \mathcal{O}_{Q_3}(2p^n - 3)$. Finally,

$$(27) \quad \text{Ext}^i(D^{p^n-2}\mathcal{U}_2(-1), F^{n*}F_*^n S^{p^n}\mathcal{U}_2^* \otimes \mathcal{O}_{Q_3}(2p^n - 3)) = H^i(Q_3, F^{n*}F_*^n S^{p^n}\mathcal{U}_2^* \otimes S^{p^n-2}\mathcal{U}_2^*(2p^n - 2)),$$

and there is an isomorphism of cohomology groups (Corollary 2.1, [11]):

$$(28) \quad \begin{aligned} H^i(Q_3, F^{n*}F_*^n S^{p^n}\mathcal{U}_2^* \otimes S^{p^n-2}\mathcal{U}_2^*(2p^n - 2)) &= \\ = H^i(Q_3 \times Q_3, (F^n \times F^n)^*(i_*\mathcal{O}_\Delta) \otimes (S^{p^n}\mathcal{U}_2^* \boxtimes S^{p^n-2}\mathcal{U}_2^*(2p^n - 2))). \end{aligned}$$

Apply $F^{n*} \times F^{n*}$ to the resolution (15). Denote C^\bullet the complex, whose terms are $C^j = F^{n*}\Psi_{-j} \boxtimes F^{n*}\mathcal{O}_{Q_3}(j)$ for $j = -2, -1, 0$ and $C^{-3} = F^{n*}\mathcal{U}_2 \boxtimes F^{n*}\mathcal{U}_2(-2)$. Tensor C^\bullet with the bundle $S^{p^n}\mathcal{U}_2^* \boxtimes S^{p^n-2}\mathcal{U}_2^*(2p^n - 2)$. Then the complex $C^\bullet \otimes (S^{p^n}\mathcal{U}_2^* \boxtimes S^{p^n-2}\mathcal{U}_2^*(2p^n - 2))$ computes the cohomology group in the right hand side of (28).

Lemma 3.4. $H^i(Q_3 \times Q_3, C^j \otimes (S^{p^n}\mathcal{U}_2^* \boxtimes S^{p^n-2}\mathcal{U}_2^*(2p^n - 2))) = 0$ for $i > -j$ and $n \geq 1$.

Clearly, this implies $H^i(Q_3 \times Q_3, (F^n \times F^n)^*(i_*\mathcal{O}_\Delta) \otimes (S^{p^n}\mathcal{U}_2^* \boxtimes S^{p^n-2}\mathcal{U}_2^*(2p^n - 2))) = 0$ for $i > 0$ and $n \geq 1$, and hence Lemma 3.3. The proof of Lemma 3.4 is broken up into a series of propositions below. \square

Proposition 3.2. $H^i(Q_3 \times Q_3, C^j \otimes (S^{p^n}\mathcal{U}_2^* \boxtimes S^{p^n-2}\mathcal{U}_2^*(2p^n - 2))) = 0$ for $i > -j$, where $j = -1, 0$ and $n \geq 1$.

Proof. Indeed, $S^k\mathcal{U}_2^* = R^\bullet q_* \pi^* \mathcal{O}_{\mathbb{P}^3}(k)$ for $k \geq 0$, and $S^{p^n-2}\mathcal{U}_2^*(2p^n - 2) = R^\bullet q_* \pi^* \mathcal{O}_{\mathbb{P}^3}(p^n - 2) \otimes q^* \mathcal{O}_{Q_3}(2p^n - 2)$. Both line bundles $\pi^* \mathcal{O}_{\mathbb{P}^3}(p^n)$ and $\pi^* \mathcal{O}_{\mathbb{P}^3}(p^n - 2) \otimes q^* \mathcal{O}_{Q_3}(2p^n - 2)$ are effective, so using the projection formula, the Kempf vanishing and the Künneth formula, we see immediately that

$$(29) \quad H^i(Q_3 \times Q_3, S^{p^n}\mathcal{U}_2^* \boxtimes S^{p^n-2}\mathcal{U}_2^*(2p^n - 2)) = 0$$

for $i > 0$. Further, the bundle $\Psi_1 = \Omega_{\mathbb{P}^4}^1(1) \otimes \mathcal{O}_{Q_3}$ has a resolution:

$$(30) \quad 0 \rightarrow \Omega_{\mathbb{P}^4}^1(1) \otimes \mathcal{O}_{Q_3} \rightarrow W^* \otimes \mathcal{O}_{Q_3} \rightarrow \mathcal{O}_{Q_3}(1) \rightarrow 0,$$

Tensoring this sequence with $S^{p^n-2}\mathcal{U}_2^*$ and using once again the Kempf vanishing and the Künneth formula we get:

$$(31) \quad H^i(Q_3 \times Q_3, (F^{n*}\Psi_1 \otimes S^{p^n}\mathcal{U}_2^*) \boxtimes S^{p^n-2}\mathcal{U}_2^*(p^n - 2)) = 0.$$

for $i > 1$. \square

Proposition 3.3. $H^i(Q_3 \times Q_3, (F^{n*}\Psi_2 \otimes S^{p^n}\mathcal{U}_2^*) \boxtimes S^{p^n-2}\mathcal{U}_2^*(-2)) = 0$ for $i > 2$ and $n \geq 1$.

Proof. Propositions 4.1 and 4.2 below ensure that for $n \geq 1$ one has $H^i(Q_3, S^{p^n-2}\mathcal{U}^*(-2)) = 0$ for $i \neq 1$ and $H^i(Q_3, F^{n*}\Psi_2 \otimes S^{p^n}\mathcal{U}_2^*) = 0$ for $i > 1$. The Künneth formula finishes the proof. \square

Proposition 3.4. $H^i(Q_3 \times Q_3, (F^{n*}\mathcal{U}_2 \otimes S^{p^n}\mathcal{U}_2^*) \boxtimes (F^{n*}\mathcal{U}_2 \otimes S^{p^n-2}\mathcal{U}_2^*(-2))) = 0$ for $i > 3$ and $n \geq 1$.

Proof. Similarly to the above lemma, Propositions 4.3 and 4.4 below imply that for $n \geq 1$ one has $H^i(Q_3, F^{n*}\mathcal{U}_2 \otimes S^{p^n}\mathcal{U}_2^*) = 0$ for $i > 1$ and $H^i(Q_3, F^{n*}\mathcal{U}_2 \otimes S^{p^n-2}\mathcal{U}_2^*(-2)) = 0$ for $i > 2$. We are done by Künneth. \square

4. End of the proof

Proposition 4.1. $H^i(Q_3, S^{p^n-2}\mathcal{U}_2^*(-2)) = 0$ for $i \neq 1$ and $n \geq 1$.

Proof. One has $H^i(Q_3, S^{p^n-2}\mathcal{U}_2^*(-2)) = H^i(G/B, \pi^*\mathcal{O}_{\mathbb{P}^3}(p^n-2) \otimes q^*\mathcal{O}_{Q_3}(-2))$. Recall that $\omega_{G/B} = \pi^*\mathcal{O}_{\mathbb{P}^3}(-2) \otimes q^*\mathcal{O}_{Q_3}(-2)$. By Serre duality one has

$$(32) \quad H^i(G/B, \pi^*\mathcal{O}_{\mathbb{P}^3}(p^n-2) \otimes q^*\mathcal{O}_{Q_3}(-2)) = H^{4-i}(G/B, \pi^*\mathcal{O}_{\mathbb{P}^3}(-p^n))^*,$$

and the last group is isomorphic to $H^{4-i}(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-p^n))^*$ that can be non-zero only if $i = 1$. \square

Proposition 4.2. $H^i(Q_3, F^{n*}\Psi_2 \otimes S^{p^n}\mathcal{U}_2^*) = 0$ for $i > 1$ and $n \geq 1$.

Proof. Consider the sequence:

$$(33) \quad 0 \rightarrow \Omega_{\mathbb{P}^4}^2(2) \otimes \mathcal{O}_{Q_3} \rightarrow \wedge^2 V^* \otimes \mathcal{O}_{Q_3} \rightarrow \Omega_{\mathbb{P}^4}^1(2) \otimes \mathcal{O}_{Q_3} \rightarrow 0.$$

Tensor it with $S^{p^n}\mathcal{U}_2^*$. Since $H^i(Q_3, S^{p^n}\mathcal{U}_2^*) = H^i(G/B, \pi^*\mathcal{O}_{\mathbb{P}^3}(p^n)) = 0$ for $i > 0$, we see that the statement will follow if we show $H^i(Q_3, F^{n*}\Omega_{\mathbb{P}^4}^1(2) \otimes \mathcal{O}_{Q_3} \otimes S^{p^n}\mathcal{U}_2^*) = 0$ for $i > 0$. Recall that $Q_3 \subset \mathbb{P}^4 = \mathbb{P}(W)$. Consider the adjunction sequence tensored with $\mathcal{O}_{Q_3}(-1)$:

$$(34) \quad 0 \rightarrow \mathcal{T}_{Q_3}(-1) \rightarrow \mathcal{T}_{\mathbb{P}^4} \otimes \mathcal{O}_{Q_3}(-1) \rightarrow \mathcal{O}_{Q_3}(1) \rightarrow 0.$$

Recall that if the characteristic p is odd then the bundle $\mathcal{T}_{Q_3}(-1)$ is self-dual, that is $\mathcal{T}_{Q_3}(-1) = \Omega_{Q_3}^1(1)$ on Q_3 (see, for instance, [10], Lemma 4.1). Dualizing the above sequence and tensoring it then with $\mathcal{O}_{Q_3}(1)$, we get:

$$(35) \quad 0 \rightarrow \mathcal{O}_{Q_3} \rightarrow \Omega_{\mathbb{P}^4}^1(2) \otimes \mathcal{O}_{Q_3} \rightarrow \mathcal{T}_{Q_3} \rightarrow 0.$$

Consequently, the statement will follow from $H^i(Q_3, F^{n*}\mathcal{T}_{Q_3} \otimes S^{p^n}\mathcal{U}_2^*) = 0$ for $i > 0$. Since p is odd, one has $\mathcal{T}_{Q_3} = S^2\mathcal{U}_2^*$. Consider the universal exact sequence:

$$(36) \quad 0 \rightarrow \mathcal{U}_2 \rightarrow V \otimes \mathcal{O}_{Q_3} \rightarrow \mathcal{U}_2^* \rightarrow 0.$$

Recall that $\det \mathcal{U}_2 = \mathcal{O}_{Q_3}(-1)$. Taking the symmetric square of this sequence and then applying the functor F^{n*} , we get:

$$(37) \quad 0 \rightarrow \mathcal{O}_{Q_3}(-p^n) \rightarrow F^{n*}\mathcal{U}_2 \otimes F^{n*}V \rightarrow F^{n*}S^2V \otimes \mathcal{O}_{Q_3} \rightarrow F^{n*}S^2\mathcal{U}_2^* \rightarrow 0.$$

Tensor it with $S^{p^n}\mathcal{U}_2^*$. Proposition 4.3 below states that $H^i(Q_3, S^{p^n}\mathcal{U}_2^* \otimes F^{n*}\mathcal{U}_2) = 0$ for $i > 1$ and $n \geq 1$. It is sufficient therefore to show that $H^3(Q_3, S^{p^n}\mathcal{U}_2^* \otimes \mathcal{O}_{Q_3}(-p^n)) = 0$, or, equivalently, that $H^3(G/B, \mathcal{O}_{\pi}(-p^n)) = 0$. Indeed, there is an isomorphism of line bundles:

$$(38) \quad q^*\mathcal{O}_{Q_3}(p^n) = \mathcal{O}_{\pi}(p^n) \otimes \pi^*\mathcal{O}_{\mathbb{P}^3}(p^n).$$

Hence, by the projection formula:

$$(39) \quad H^3(Q_3, S^{p^n} \mathcal{U}_2^* \otimes \mathcal{O}_{Q_3}(-p^n)) = H^3(Q_3, q_*(\pi^* \mathcal{O}_{\mathbb{P}^3}(p^n) \otimes q^* \mathcal{O}_{Q_3}(-p^n)) = H^3(\mathbf{G}/\mathbf{B}, \mathcal{O}_\pi(-p^n)).$$

Considering the sequence (18) and applying Proposition 3.1 we get the statement. \square

Proposition 4.3. $H^i(Q_3, S^{p^n} \mathcal{U}_2^* \otimes F^{n*} \mathcal{U}_2) = 0$ for $i > 1$ and $n \geq 1$.

Proof. Apply F^{n*} to the sequence (36):

$$(40) \quad 0 \rightarrow F^{n*} \mathcal{U}_2 \rightarrow F^{n*} \mathbf{V} \otimes \mathcal{O}_{Q_3} \rightarrow F^{n*} \mathcal{U}_2^* \rightarrow 0.$$

Tensoring the sequence (40) with $S^{p^n} \mathcal{U}_2^*$, we obtain:

$$(41) \quad 0 \rightarrow F^{n*} \mathcal{U}_2 \otimes S^{p^n} \mathcal{U}_2^* \rightarrow F^{n*} \mathbf{V} \otimes S^{p^n} \mathcal{U}_2^* \rightarrow F^{n*} \mathcal{U}_2^* \otimes S^{p^n} \mathcal{U}_2^* \rightarrow 0$$

We saw above that the higher cohomology groups of $S^{p^n} \mathcal{U}_2^*$ vanish. Hence, from the long exact cohomology sequence it is sufficient to show that $H^i(Q_3, S^{p^n} \mathcal{U}_2^* \otimes F^{n*} \mathcal{U}_2^*) = 0$ for $i > 0$. Take the dual to the sequence (18):

$$(42) \quad 0 \rightarrow \mathcal{O}_\pi(p^n) \rightarrow q^* F^{n*} \mathcal{U}_2^* \rightarrow \pi^* \mathcal{O}_{\mathbb{P}^3}(p^n) \rightarrow 0.$$

Tensor this sequence with $\pi^* \mathcal{O}_{\mathbb{P}^3}(p^n)$:

$$(43) \quad 0 \rightarrow \mathcal{O}_\pi(p^n) \otimes \pi^* \mathcal{O}_{\mathbb{P}^3}(p^n) \rightarrow \pi^* \mathcal{O}_{\mathbb{P}^3}(p^n) \otimes q^* F^{n*} \mathcal{U}_2^* \rightarrow \pi^* \mathcal{O}_{\mathbb{P}^3}(2p^n) \rightarrow 0.$$

Using the isomorphism (38) we get:

$$(44) \quad 0 \rightarrow q^* \mathcal{O}_{Q_3}(p^n) \rightarrow \pi^* \mathcal{O}_{\mathbb{P}^3}(p^n) \otimes q^* F^{n*} \mathcal{U}_2^* \rightarrow \pi^* \mathcal{O}_{\mathbb{P}^3}(2p^n) \rightarrow 0.$$

Applying to this sequence the functor q_* , and using the projection formula and the isomorphism $R^\bullet q_* \pi^* \mathcal{O}_{\mathbb{P}^3}(k) = S^k \mathcal{U}_2^*$ for $k \geq 0$, we obtain:

$$(45) \quad 0 \rightarrow \mathcal{O}_{Q_3}(p^n) \rightarrow S^{p^n} \mathcal{U}_2^* \otimes F^{n*} \mathcal{U}_2^* \rightarrow S^{2p^n} \mathcal{U}_2^* \rightarrow 0.$$

The leftmost and rightmost terms of the above sequence have vanishing higher cohomology, hence the statement of the lemma. \square

Proposition 4.4. $H^3(Q_3, S^{p^n-2} \mathcal{U}_2^*(-2) \otimes F^{n*} \mathcal{U}_2) = 0$ for $n \geq 1$.

Proof. One has $H^i(Q_3, S^{p^n-2} \mathcal{U}_2^*(-2) \otimes F^{n*} \mathcal{U}_2) = H^i(\mathbf{G}/\mathbf{B}, \pi^* \mathcal{O}_{\mathbb{P}^3}(p^n-2) \otimes q^* \mathcal{O}_{Q_3}(-2) \otimes q^* F^{n*} \mathcal{U}_2)$. By Serre duality we get:

$$(46) \quad H^3(\mathbf{G}/\mathbf{B}, \pi^* \mathcal{O}_{\mathbb{P}^3}(p^n-2) \otimes q^* \mathcal{O}_{Q_3}(-2) \otimes q^* F^{n*} \mathcal{U}_2) = H^1(\mathbf{G}/\mathbf{B}, \pi^* \mathcal{O}_{\mathbb{P}^3}(-p^n) \otimes q^* F^{n*} \mathcal{U}_2^*)^*.$$

Tensor the sequence (40) with $\pi^* \mathcal{O}_{\mathbb{P}^3}(-p^n)$. One has:

$$(47) \quad \begin{aligned} \dots \rightarrow H^1(\mathbf{G}/\mathbf{B}, F^{n*} \mathbf{V} \otimes \pi^* \mathcal{O}_{\mathbb{P}^3}(-p^n)) &\rightarrow H^1(\mathbf{G}/\mathbf{B}, \pi^* \mathcal{O}_{\mathbb{P}^3}(-p^n) \otimes q^* F^{n*} \mathcal{U}_2^*) \rightarrow \\ &\rightarrow H^2(\mathbf{G}/\mathbf{B}, \pi^* \mathcal{O}_{\mathbb{P}^3}(-p^n) \otimes q^* F^{n*} \mathcal{U}_2) \rightarrow \dots \end{aligned}$$

Clearly, $H^1(\mathbf{G}/\mathbf{B}, F^{n*} \mathbf{V} \otimes \pi^* \mathcal{O}_{\mathbb{P}^3}(-p^n)) = H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-p^n)) \otimes F^{n*} \mathbf{V} = 0$. Let us show that $H^2(\mathbf{G}/\mathbf{B}, \pi^* \mathcal{O}_{\mathbb{P}^3}(-p^n) \otimes q^* F^{n*} \mathcal{U}_2) = 0$. Indeed, tensoring the sequence (18) with $\pi^* \mathcal{O}_{\mathbb{P}^3}(-p^n)$ we see that $H^i(\mathbf{G}/\mathbf{B}, \pi^* \mathcal{O}_{\mathbb{P}^3}(-p^n) \otimes q^* F^{n*} \mathcal{U}_2) = 0$ if $i \neq 3$, hence the statement. \square

REFERENCES

- [1] H.H.Andersen, Cohomology of line bundles, Algebraic groups and homogeneous spaces, Tata Inst. Fund. Res. Stud. Math., (2007), 13–36
- [2] H. H. Andersen, M. Kaneda, On the D-affinity of the flag variety in type B_2 , Manuscripta Math. 103 (2000), 393–399
- [3] A. Beilinson, J. Bernstein, “Localisation de \mathfrak{g} -modules”, C.R.Acad. Sci. Paris, Ser. I 292 (1981), 15–18
- [4] B. Haastert, Über Differentialoperatoren und \mathbb{D} -Moduln in positiver Charakteristik, Manuscripta Math., 58 (1987), no. 4, 385–415
- [5] J.C.Jantzen, Representations of algebraic groups, 2nd edition, Mathematical Surveys and Monographs, 107, (2003)
- [6] M. Kashiwara, N. Lauritzen, Local cohomology and \mathcal{D} -affinity in positive characteristic, C. R. Acad. Sci. Paris, Ser. I, 335 (2002), 993–996
- [7] A. Langer, D-affinity and Frobenius morphism on quadrics, Int. Math. Res. Not., 2008, no. 1, Art. ID rnm 145, 26 pp.
- [8] D. Orlov, Projective bundles, monoidal transformations, and derived categories of coherent sheaves, (Russian) Izv. Ross. Akad. Nauk Ser. Mat. 56 (1992), no. 4, 852–862; translation in Russian Acad. Sci. Izv. Math. 41 (1993), no. 1, 133–141
- [9] A. Samokhin, On the D-affinity of quadrics in positive characteristic, C. R. Math. Acad. Sci. Paris 344 (2007), no. 6, 377–382.
- [10] A. Samokhin, A vanishing theorem for sheaves of small differential operators in positive characteristic, arXiv:0707.0913, to appear in Transformation Groups
- [11] A. Samokhin, Tilting bundles via the Frobenius morphism, arXiv:0904.1235v1

INSTITUTE FOR INFORMATION TRANSMISSION PROBLEMS, MOSCOW, RUSSIA

E-mail address: `alexander.samokhin@gmail.com`